On the energy transported by exact plane gravitational-wave solutions

Yuri N. Obukhov*

Instituto de Física Teórica, UNESP-Universidade Estadual Paulista,
Rua Dr. Bento Teobaldo Ferraz 271, 01140-070 São Paulo, Brazil and
Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia

J. G. Pereira†

Instituto de Física Teórica, UNESP-Universidade Estadual Paulista,
Rua Dr. Bento Teobaldo Ferraz 271, 01140-070 São Paulo, Brazil

Guillermo F. Rubilar‡

Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile

Abstract

The energy and momentum transported by *exact* plane gravitational-wave solutions of Einstein equations are computed using the teleparallel equivalent formulation of Einstein's theory. It is shown that these waves transport neither energy nor momentum. A comparison with the usual *linear* plane gravitational-waves solution of the linearized Einstein equation is presented.

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*Electronic address: yo@ift.unesp.br

†Electronic address: jpereira@ift.unesp.br

‡Electronic address: grubilar@udec.cl

I. INTRODUCTION

Although there are compelling indirect experimental evidences [1] of the existence of gravitational waves [2], they have not yet been directly detected. These evidences, it is important to remark, do not give a clue on the form and properties of the gravitational waves. Since the potential astrophysical sources of gravitational waves are at enormous distances from us, the amplitude of a wave when reaching a detector on Earth should be so small that the linearized theory is usually assumed to provide a sufficiently accurate description of these waves. In other words, to comply with the idea that the radiative solutions satisfy a linear wave equation, one should assume that these waves do not carry enough energy and momentum to affect their own plane-wave form [3].

The search for gravitational waves relies strongly on the assumptions above. The lack of direct observational evidences, however, may question these assumptions. In particular, it is well known that a field can transport its own source only via an essentially nonlinear process [4]. This point is the origin of a long-standing discussion on the energy and momentum transported by gravitational waves [5].

The study of gravitational waves in general relativity has a long and rich history [6, 7, 8, 9]. The discussion of the possible generalizations of such solutions revealed the exact wave solutions in Poincaré gauge gravity [10, 11, 12, 13, 14, 15], in teleparallel gravity [16], in generalized Einstein theories [17, 18], in metric-affine theory of gravity [19, 20, 21, 22, 23, 24, 25], in supergravity [26, 27, 28, 29, 30], as well as, more recently, in superstring theories [31, 32, 33]. Some attention has also been paid to the higher-dimensional generalizations of the gravitational wave solutions [34, 35, 36].

On the other hand, plane-fronted gravitational waves represent an important class of exact solutions of Einstein's equation [37, 38, 39]. In this paper we consider the question of the energy-momentum transported by such plane-wave solutions. Of course, any exact gravitational wave solution is highly idealized, and is thus probably more of academic interest [40]. Nevertheless, we expect that our study may shed some new light on that question, as well as to develop a further understanding of the nonlinear regime of gravitation. Our general notations are as in [41]. In particular, we use the Latin indices $i, j, \ldots$ for local holonomic spacetime coordinates and the Greek indices $\alpha, \beta, \ldots$ label (co)frame components. Particular frame components are denoted by hats, $\hat{0}, \hat{1}$, etc. As usual, the exterior product is denoted

by $\wedge$, while the interior product of a vector ξ and a p-form Ψ is denoted by $\xi\rfloor\Psi$. The vector basis dual to the frame 1-forms ϑ^α is denoted by e_α and they satisfy $e_\alpha\rfloor\vartheta^\beta = \delta^\beta_\alpha$. Using local coordinates x^i, we have $\vartheta^\alpha = h^\alpha_i dx^i$ and $e_\alpha = h^i_\alpha \partial_i$. We define the volume 4-form by $\eta := \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}$. Furthermore, with the help of the interior product we define $\eta_\alpha := e_\alpha\rfloor\eta$, $\eta_{\alpha\beta} := e_\beta\rfloor\eta_\alpha$, $\eta_{\alpha\beta\gamma} := e_\gamma\rfloor\eta_{\alpha\beta}$ which are bases for 3-, 2- and 1-forms respectively. Finally, $\eta_{\alpha\beta\mu\nu} = e_\nu\rfloor\eta_{\alpha\beta\mu}$ is the Levi-Civita tensor density. The η-forms satisfy the useful identities:

$$\vartheta^\beta \wedge \eta_\alpha = \delta^\beta_\alpha \eta, \tag{1.1}$$

$$\vartheta^\beta \wedge \eta_{\mu\nu} = \delta^\beta_\nu \eta_\mu - \delta^\beta_\mu \eta_\nu, \tag{1.2}$$

$$\vartheta^\beta \wedge \eta_{\alpha\mu\nu} = \delta^\beta_\alpha \eta_{\mu\nu} + \delta^\beta_\mu \eta_{\nu\alpha} + \delta^\beta_\nu \eta_{\alpha\mu}, \tag{1.3}$$

$$\vartheta^\beta \wedge \eta_{\alpha\gamma\mu\nu} = \delta^\beta_\nu \eta_{\alpha\gamma\mu} - \delta^\beta_\mu \eta_{\alpha\gamma\nu} + \delta^\beta_\gamma \eta_{\alpha\mu\nu} - \delta^\beta_\alpha \eta_{\gamma\mu\nu}. \tag{1.4}$$

The line element $ds^2 = g_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$ is defined by the spacetime metric $g_{\alpha\beta}$ of signature $(+,-,-,-)$.

II. TELEPARALLEL GRAVITY

The teleparallel approach is based on the gauge theory of translations, and the coframe $\vartheta^\alpha = h^\alpha_i dx^i$ (tetrad) plays the role of the corresponding gravitational potential. Einstein's general relativity theory can be reformulated as the teleparallel theory. Geometrically, one can view the teleparallel gravity as a special (degenerate) case of the metric-affine gravity in which the coframe ϑ^α and the local Lorentz connection $\Gamma_\alpha{}^\beta$ are subject to the distant parallelism constraint $R_\alpha{}^\beta = 0$ [42]. The torsion 2-form

$$T^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta, \tag{2.1}$$

arises as the gravitational gauge field strength. In the so-called teleparallel equivalent gravity model, the Lagrangian reads:

$$V = -\frac{1}{2\kappa} T^\alpha \wedge {}^\star\left(T_\alpha - \vartheta^\alpha \wedge e_\beta\rfloor T^\beta - \frac{1}{2}\, e^\alpha\rfloor(\vartheta^\beta \wedge T_\beta)\right). \tag{2.2}$$

Here $\kappa = 8\pi G/c^3$, and * denotes the Hodge duality operator in the metric $g_{\alpha\beta}$. The latter is assumed to be the flat Minkowski metric $g_{\alpha\beta} = o_{\alpha\beta} := \mathrm{diag}(+1,-1,-1,-1)$, and we use it to raise and lower local frame (Greek) indices.

The teleparallel field equations are obtained from the variation of the total action with respect to the coframe,

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \tag{2.3}$$

where $DH_\alpha = dH_\alpha - \Gamma_\alpha^{\ \beta} \wedge H_\beta$ denotes the covariant exterior derivative. The translational momentum and the canonical energy-momentum are, respectively:

$$\begin{aligned} H_\alpha = -\frac{\partial V}{\partial T^\alpha} &= \frac{1}{\kappa}{}^\star\Big(T_\alpha - \vartheta_\alpha \wedge e_\beta \rfloor T^\beta - \frac{1}{2}\, e_\alpha \rfloor (\vartheta^\beta \wedge T_\beta)\Big)\,, & (2.4)\\ E_\alpha = \frac{\partial V}{\partial \vartheta^\alpha} &= e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) \wedge H_\beta. & (2.5) \end{aligned}$$

The Lagrangian (2.2) can then be recast as

$$V = -\frac{1}{2}\, T^\alpha \wedge H_\alpha. \tag{2.6}$$

It should be mentioned that the resulting model is degenerate, from the metric-affine viewpoint, because the variational derivatives of the action with the respect to the metric and connection are trivial. This means that the field equations are satisfied for any $\Gamma_\alpha{}^\beta$. However, the presence of the connection field plays an important role that may be characterized as a *regularization*. The latter is twofold.

First of all, teleparallel gravity becomes explicitly covariant under the local Lorentz transformations of the coframe. In particular, the Lagrangian (2.2) is invariant under the change of variables

$$\vartheta'^\alpha = L^\alpha{}_\beta \vartheta^\beta, \qquad \Gamma'^{\ \beta}_\alpha = (L^{-1})^\mu{}_\alpha \Gamma_\mu{}^\nu L^\beta{}_\nu + L^\beta{}_\gamma d(L^{-1})^\gamma{}_\alpha, \tag{2.7}$$

with $L^\alpha{}_\beta(x) \in SO(1,3)$. In contrast, for the pure tetrad gravity, which is obtained when we put $\Gamma_\alpha{}^\beta = 0$, the Lagrangian is only quasi-invariant—it changes by a total divergence. The connection 1-form $\Gamma_\alpha{}^\beta$ can be decomposed into the Riemannian and post-Riemannian parts as

$$\Gamma_\alpha{}^\beta = \tilde{\Gamma}_\alpha{}^\beta - K_\alpha{}^\beta. \tag{2.8}$$

Here $\tilde{\Gamma}_\alpha{}^\beta$ is the purely Riemannian connection and $K_\alpha{}^\beta$ is the contortion 1-form which is related to the torsion through the identity

$$T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta. \tag{2.9}$$

Then one can show that due to geometric identities [43], the gauge momentum (2.4) can be written as

$$H_\alpha = \frac{1}{2\kappa}\, K^{\mu\nu} \wedge \eta_{\alpha\mu\nu}. \tag{2.10}$$

A second, and even more important property of the teleparallel framework is that the Weitzenböck connection actually represents inertial effects that arise due to the choice of the reference system [44]. The inertial contributions in many cases yield unphysical results for the total energy of the system, producing either trivial or divergent answers. The teleparallel connection acts as a regularizing tool which helps to subtract the inertial effects without distorting the true gravitational contribution [45].

III. ENERGY-MOMENTUM CONSERVATION

We begin by rewriting the field equation (2.3) in the Maxwell-type form:

$$DH_\alpha = E_\alpha + \Sigma_\alpha. \tag{3.1}$$

The analogy with the electromagnetism is obvious. The Maxwell 2-form $F = dA$ represents the gauge field strength of the electromagnetic potential 1-form A. From the Lagrangian $V(F)$, the 2-form of the electromagnetic excitations is defined by $H = -\partial V/\partial F$, and the field equation reads $dH = J$, where J is the 3-form of the electric current density of matter. In view of the nilpotency of the exterior differential, $dd \equiv 0$, the Maxwell equation yields the conservation law of the electric current, $dJ = 0$.

In contrast to electrodynamics, gravity is fundamentally nonlinear. It is more akin to a Yang-Mills theory, though the "internal" index of the gauge field potential 1-form ϑ^α is not really internal, but of spacetime nature. The gauge field strength 2-form $T^\alpha = D\vartheta^\alpha$ is now defined by the covariant derivative of the potential (compare with $F = dA$). The gravitational field excitation 2-form H_α is introduced by (2.4), in a direct analogy to the Maxwell theory (recall $H = -\partial V/\partial F$). Finally, we observe that as compared to the Maxwell field equation $dH = J$, the gravitational field equation (3.1) contains now the covariant derivative D, and in addition, the right-hand side is represented by a modified current 3-form, $E_\alpha + \Sigma_\alpha$. The last term is the energy-momentum of matter, and we naturally conclude that the 3-form E_α describes the *energy-momentum current of the gravitational field*. Its presence in the right-hand side of the field equation (3.1) reflects the self-interacting nature of the gravitational field.

We can complete the analogy with electrodynamics by deriving the corresponding conservation law. Indeed, since $DD \equiv 0$ for the teleparallel connection, (3.1) tells us that the sum

of the energy-momentum currents of gravity and matter, $E_\alpha + \Sigma_\alpha$, is covariantly conserved [44],

$$D(E_\alpha + \Sigma_\alpha) = 0. \tag{3.2}$$

This covariant conservation law is consistent with the covariant transformation properties of the currents E_α and Σ_α.

One can rewrite the conservation of energy-momentum in terms of the ordinary derivatives. Using the explicit expression $DH_\alpha = dH_\alpha - \Gamma_\alpha{}^\beta \wedge H_\beta$, the field equation (2.3) and (3.1) can be recast in an alternative form

$$dH_\alpha = \mathcal{E}_\alpha + \Sigma_\alpha, \tag{3.3}$$

where $\mathcal{E}_\alpha = E_\alpha + \Gamma_\alpha{}^\beta \wedge H_\beta$. Accordingly, Eq. (3.3) yields a usual conservation law with the ordinary derivative,

$$d(\mathcal{E}_\alpha + \Sigma_\alpha) = 0. \tag{3.4}$$

The 3-form E_α describes the gravitational energy-momentum in a covariant way, whereas the 3-form $\mathcal{E}_\alpha$ is a non-covariant object. In terms of components, $\mathcal{E}_\alpha$ gives rise to the energy-momentum pseudotensor. It is worthwhile to note that H_α thus plays a role of the energy-momentum superpotential both for the total covariant energy-momentum current $(E_\alpha + \Sigma_\alpha)$ and for the noncovariant current $(\mathcal{E}_\alpha + \Sigma_\alpha)$.

The η-forms (defined above) serve as the basis of the spaces of forms of different rank, and when we expand the above objects with respect to the η-forms, the usual tensor formulation is recovered. Explicitly,

$$H_\alpha = \frac{1}{\kappa}\, S_\alpha{}^{\mu\nu}\, \eta_{\mu\nu}. \tag{3.5}$$

Here $S_\alpha{}^{\mu\nu} = -S_\alpha{}^{\nu\mu}$ is constructed from the contortion tensor in the usual way.

Analogously, we have explicitly for the gravitational energy-momentum

$$E_\alpha = \frac{1}{2}\Big[(e_\alpha\rfloor T^\beta) \wedge H_\beta - T^\beta \wedge (e_\alpha\rfloor H_\beta)\Big]\,. \tag{3.6}$$

Substituting here (3.5) and $T^\alpha = \frac{1}{2}\, T_{\mu\nu}{}^\alpha\, \vartheta^\mu \wedge \vartheta^\nu$ and using Eqs. (1.2)-(1.4), we find [44]

$$E_\alpha = t_\alpha{}^\beta\, \eta_\beta, \qquad t_\alpha{}^\beta = \frac{1}{2\kappa}\left(4T_{\alpha\nu}{}^\lambda S_\lambda{}^{\beta\nu} - T_{\mu\nu}{}^\lambda S_\lambda{}^{\mu\nu}\, \delta_\alpha^\beta\right). \tag{3.7}$$

Similarly, we find

$$\mathcal{E}_\alpha = j_\alpha{}^\beta\, \eta_\beta, \qquad j_\alpha{}^\beta = \frac{1}{2\kappa}\left(4T_{\alpha\nu}{}^\lambda S_\lambda{}^{\beta\nu} - T_{\mu\nu}{}^\lambda S_\lambda{}^{\mu\nu}\, \delta_\alpha^\beta + 4\Gamma_{\nu\alpha}{}^\lambda\, S_\lambda{}^{\beta\nu}\right). \tag{3.8}$$

Taking into account the analogous expansion of the matter energy-momentum, $\Sigma_\alpha = \Sigma_\alpha{}^\beta\,\eta_\beta$, that introduces the tensor of energy-momentum, and using (3.5) and (3.7), we easily recover the field equation in tensor language (used, for example, in [46]). Note that the conservation laws (3.2) and (3.4) coincide when we put $\Gamma_\alpha{}^\beta = 0$. The last term in (3.8) then disappears, whereas the torsion reduces to the anholonomity 2-form, $T^\alpha = F^\alpha = d\vartheta^\alpha$ [47, 48].

IV. EXACT PLANE WAVE

In the local coordinates (σ, ρ, y, z), the plane-fronted gravitational wave [6, 7] is described by the coframe components:

$$\vartheta^{\widehat{0}} = (1 + h/4)d\sigma + d\rho, \qquad \vartheta^{\widehat{1}} = (1 - h/4)d\sigma - d\rho, \qquad \vartheta^{\widehat{2}} = dy, \qquad \vartheta^{\widehat{3}} = dz. \tag{4.1}$$

When the function $h(\sigma, y, z)$ satisfies the two-dimensional Laplace equation,

$$\frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0, \tag{4.2}$$

the coframe (4.1) represents an exact solution of Einstein's field equation in vacuum. In particular, one usually chooses $h = yz\,u(\sigma)$ or $h = (y^2 - z^2)v(\sigma)$, where u, v are arbitrary functions of σ.

This configuration describes a plane-fronted gravitational wave that is characterized by the wave covector $k = k_\alpha \vartheta^\alpha = 2d\sigma$, so that the wave fronts are represented by the plane surfaces of constant σ. The coordinate ρ is an affine parameter along the wave vector of the null geodesic. As we see, the tetrad components of the wave covector read $k_\alpha = (1, 1, 0, 0)$, and hence $k^\alpha = (1, -1, 0, 0)$. Obviously, $k^\alpha k_\alpha = 0$.

V. ENERGY TRANSPORTED BY AN EXACT PLANE WAVE

The coframe (4.1) becomes holonomic when gravitation is switched off. In technical terms this means that its anholonomy is not related to inertial effects, but only to gravitation. Accordingly, we conclude that in this frame the Weitzenböck connection can be consistently put equal to zero: $\Gamma_\alpha{}^\beta = 0$. Torsion then reads

$$T^\alpha = \frac{k^\alpha}{4}\, d\sigma \wedge dh. \tag{5.1}$$

In this case, Eq. (3.5) yields

$$H_\alpha = -\frac{k_\alpha}{4}\, d\sigma \wedge \underline{\star} dh, \tag{5.2}$$

where $\underline{\star}$ denotes the Hodge operator in the 2-dimensional Euclidean plane (y, z), namely, $\underline{\star}(a\,dy + b\,dz) = a\,dz - b\,dy$. Hence, from Eq. (3.6) we see that

$$E_\alpha = 0. \tag{5.3}$$

This is a covariant result that does not depend on the choice of the reference system. When we choose a different reference system, the corresponding coframe is related by a Lorentz transformation to the original one, and the teleparallel connection in this system is computed from (2.7). It will be nontrivial, in general, thus reflecting the possible non-inertiality of the reference system. Since under the Lorentz transformation (2.7) the 3-form of the gravitational energy-momentum transforms covariantly, $E'_\alpha = (L^{-1})^\beta{}_\alpha E_\beta$, it will be zero in all reference systems.

Earlier, it was demonstrated [49] that it is possible to choose local coordinates in such a way that the pseudotensor of the energy-momentum of plane gravitational waves vanishes. Our result is, however, much stronger. Due to the use of the exterior calculus, all our computations are independent of the choice of the local coordinates. In addition, the covariance of the framework under the local Lorentz transformations makes our results independent also of the choice of the reference system. We can then conclude that the exact plane-wave solution (4.1) transports neither energy nor momentum.

VI. REMARK ON SUPERENERGY

In the studies of gravitational waves, there is a long history of the attempts to describe the energy and momentum of the pure radiation by the so called superenergy tensors; the most well-known of them is perhaps the Bel-Robinson tensor [52]. In the case of general relativity, the corresponding conserved current 3-form is defined by

$$B_{\alpha\mu\nu} = \frac{1}{2}\left[(e_\alpha\rfloor \widetilde{R}_\mu{}^\lambda) \wedge {}^\star \widetilde{R}_{\nu\lambda} - \widetilde{R}_\nu{}^\lambda \wedge (e_\alpha\rfloor {}^\star \widetilde{R}_{\mu\lambda})\right], \tag{6.1}$$

where $\widetilde{R}_\mu{}^\nu = \frac{1}{2}\, \widetilde{R}_{\alpha\beta\mu}{}^\nu\, \vartheta^\alpha \wedge \vartheta^\beta$ is the Riemannian curvature 2-form. There is an obvious similarity between (6.1) and the definition of the gravitational energy-momentum current

(3.6). It is also similar to the energy-momentum tensor of the electromagnetic field,

$$E^{(\mathrm{em})}_\alpha = \frac{1}{2}\left[(e_\alpha\rfloor F)\wedge H - F\wedge(e_\alpha\rfloor H)\right], \tag{6.2}$$

with F the Maxwell field strength 2-form, and $H = {}^\star F$ the electromagnetic excitation 2-form. Notice, however, that the superenergy carries more indices than the energy-momentum forms.

Using the explicit form (4.1) of the plane-wave metric in the Bel-Robinson tensor (6.1), we find

$$B_{\alpha\mu\nu} = |\gamma|^2\, k_\alpha k_\beta k_\mu k_\nu\, \eta^\beta. \tag{6.3}$$

Here $|\gamma|^2 := {}^\star(\gamma_\alpha\wedge{}^\star\gamma^\alpha)$, where in accordance with Ref. [36] we introduce the 1-form γ_α by $\gamma_{\hat 0} = \gamma_{\hat 1} = 0$ and $\gamma_a = -\frac{1}{2}d(e_a\rfloor dh)$, for $a = 2, 3$. This 1-form is the key element that determines the structure of the Riemannian curvature of the plane-wave solution (see the eq. (7) of [36]), and it describes the polarization properties of the wave. The factor $|\gamma|^2$ is a positive quantity, and for the specific choices of the function h mentioned above, $|\gamma|^2$ is proportional to $u^2(\sigma)$ and $v^2(\sigma)$. Thus indeed the Bel-Robinson form has a nontrivial value for the plane waves, with the structure of (6.3) resembling the expressions for the energy of the electromagnetic waves.

Unfortunately, however, the dynamical role of the Bel-Robinson superenergy, despite a number of nice properties, is still unclear. In particular, the main problem remains on the choice of the overall factor in (6.1), which is needed to obtain the correct dimension. In this respect, it is worthwhile to mention a recent paper [53] that addresses the problem of finding a dynamical framework for the Bel-Robinson superenergy.

VII. DISCUSSION AND CONCLUSION

As we have seen, the exact plane-wave solution of Einstein's equation transports neither energy nor momentum. This property appears to be quite unusual, since one could expect that a gravitational wave, like any other gravitational field configuration, should be characterized by nontrivial distribution of energy and momentum. In this sense the exact plane wave solution, although being mathematically well defined, appears to be a physical puzzle. A detailed discussion of its meaning may thus be needed. At the moment it is unclear whether such a property is shared by other exact wave solutions known in the literature,

although the fact that the plane wave configuration is simultaneously a solution of full Einstein's equation and of the linearized equation appears to be an important property that most probably yields the vanishing of the energy.

One can wonder whether the inability to transport energy and momentum is a property of this specific exact solution, or it is a general property of any linear gravitational waves. In order to get some insight into this question, a comparison with electromagnetism is quite elucidative. Let us rewrite Maxwell equation as follows

$$dH = E + J, \tag{7.1}$$

where we introduced the electric self-interaction current E along with the usual electric matter current 3-form J. As is well known, electromagnetism is linear, and since any self-current is at least quadratic in the field variables, the electromagnetic self-interaction term vanishes identically: $E = 0$. Due to this linearity, an electromagnetic wave does not transport its own source, that is, the electric charge. Of course, electromagnetic waves are able to transport energy and momentum, since neither energy nor momentum is a source of the electromagnetic field, and the energy-momentum current does not appear in the electromagnetic field equation. Accordingly, the linearity of electromagnetism does not restrict the energy-momentum current to be linear. In fact, the energy transported by an electromagnetic wave is given by the quadratic Poynting vector.

The picture changes significantly for gravitation. The crucial difference is that energy-momentum is source of gravitation, and consequently *the energy-momentum self-interaction current E_α appears explicitly in the gravitational field equations*

$$DH_\alpha = E_\alpha + \Sigma_\alpha. \tag{7.2}$$

In the linear approximation, the energy-momentum current is restricted to be linear, and consequently it vanishes in that approximation [50]. We can then say that, similarly to the fact that electromagnetic waves are unable to transport its own (electric) charge, a linear gravitational wave seems to be unable to transport its "charge", that is, the energy-momentum. Usually one takes the second-order energy-momentum current, but this is a questionable procedure because such a current appears as source only in the second-order wave equation. It should, by this reason, represent the energy-momentum transported by the second-order nonlinear gravitational wave.

Our results do not mean at all that gravitational waves cannot exist. However, they seem to support the idea that the true gravitational wave should be essentially a nonlinear physical phenomenon. The analysis of the specific example of the exact plane wave considered here shows that plane waves seem to have rather unusual properties, such as the vanishing of their energy-momentum current [51]. In the light of our observations, their physical meaning remains uncertain. Finally, we would like to add that, while technically obtained in the context of the teleparallel equivalent formulation of Einstein's theory, this result is physically meaningful, and as such it might also be true in the context of general relativity.

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